Stratifying Generalized Linear Systems by means a Derivative Feedback

M. Isabel García-Planas
Universitat Politècnica de Catalunya
Spain
E-mail: maria.isabel.garcia@upc.es

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Abstract

We consider generalized linear time invariant systems in the form $E\dot{x}(t) = Ax(t) + Bu(t)$ with $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, arising naturally in a variety of circumstances.

It is well known that the regularity condition $\det E \neq 0$ guarantees the existence and uniqueness of solutions. In this work we study necessary and sufficient conditions in order that the system can be normalized by means of a derivative feedback, that is to say conditions that ensure the existence of a control $u(t) = -V\dot{x}(t) + w(t)$ with $V \in M_{m\times n}(\mathbb{C})$ such that the matrix $E + BV$ in the closed loop system $(E + BV)\dot{x}(t) = Ax(t) + Bw(t)$ is invertible and consequently the close loop system is uniquely solvable. We also study conditions for controllability of the close loop system ensuring the existence of a feedback $w(t) = Ux(t) + \omega(t)$ with $V \in M_{m\times n}(\mathbb{C})$, such that the system $(E + BV)\dot{x}(t) = (A + BU)x(t) + B\omega(t)$ has a stable solution.

Finally, a stratification of the space of orbits of the systems that can be normalized is presented.

- Generalized linear dynamical systems,
- Singular systems,
- Daes,
- Descriptor systems,
- semi-state systems,
- ....

$$\boxed{\boxed{E\dot{x}(t) = Ax(t) + Bu(t)}} \tag{1}$$

$E, A \in M_{p\times n}(\mathbb{C}), B \in M_{n\times m}(\mathbb{C})$.

[1] L. Dai, *Singular Control Systems*, Springer, Berlin, 1989.

[2] P.Kunkel, V. Mehrmann, *A new look at pencils of matrix valued functions*, LAA 212/213 (1994) 215-248.

[3] W.C. Rheinboldt, *Differential-algebraic systems as differential equations on manifolds*, Math. Comput. 43 (1984) 473-482.

[4] S. Feldmann, G.Heining, *Partial realization for singular systems in standard form.* LAA 318 (2000), pp 127-144.

Notations:

- $M_{r\times s}(\mathbb{C})$ the space of complex matrices having r rows and s columns, and in the case which $r = s$ we write $M_r(\mathbb{C})$.

- $\mathcal{M} = \{(E, A, B) \mid E, A \in M_n(\mathbb{C}), B \in M_{n\times m}(\mathbb{C})\}$ corresponding to a generalized time-invariant linear systems

$$\boxed{E\dot{x}(t) = Ax(t) + Bu(t)} \tag{2}$$

Remark 1. when $E = I_n$ it is called standard system, and we write (A, B) for $\dot{x}(t) = Ax(t) + Bu(t)$.

If E invertible:

$$\boxed{E\dot{x}(t) = Ax(t) + Bu(t)}$$

$$\downarrow$$

$$\boxed{\dot{x} = E^{-1}Ax(t) + E^{-1}Bu(t)}$$

Definition 1. *We say that a generalized system is standardizable if the matrix E is invertible.*

Some authors call normalizable instead of standardizable

Example 1. We consider the generalized system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

The solution is:

$$\boxed{\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -\dot{u}(t) \\ -u(t) \end{pmatrix}}$$

- Then: there not exists solution with initial condition x_0 unless $x_0 = (-\dot{u}(0), -u(0))$.

If we consider $u = u_1 - V\dot{x}$ with $V = \begin{pmatrix} 1 & 0 \end{pmatrix}$ the system becomes $(E + BV)\dot{x} = Ax + Bu_1$ with $(E + BV)$ invertible.

The standard system:

$$\boxed{\dot{x} = (E + BV)^{-1}Ax + (E + BV)^{-1}Bu_1}$$

is controllable.

If E not invertible:

$\exists\, u_1(t) = u(t) + G\dot{x}(t), G \in M_{m\times n}(\mathbb{C})$
such that $E + BG$ is invertible?

if yes

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

$$\downarrow$$

$$\dot{x}(t)=(E+BG)^{-1}Ax(t)+(E+BG)^{-1}Bu(t)$$

Definition 2. *We say that a generalized system is standardizable by derivative feedback if the matrix $E+BG$ for some matrix G, is invertible.*

Remark 2. Standardizble $\implies$ Standardizable under derivative feedeback.

(It suffices to take $G = 0$)

In the sequel we will call *standardizable systems*.

Feedback and derivative feedback Equivalence

Definition 3. *(E_1, A_1, B_1) is equivalent to (E_2, A_2, B_2) if and only if there exists invertible matrices $P \in Gl(n;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, and matrices $F, G \in M_{m\times n}(\mathbb{C})$ such that*

$$\boxed{\begin{array}{l} E_2 = P^{-1}E_1P + P^{-1}BG, \\ A_2 = P^{-1}A_1P + P^{-1}BF, \\ B_2 = P^{-1}B_1R, \end{array}} \qquad (3)$$

- Matrix expression

$$\boxed{\begin{pmatrix} E_2 & A_2 & B_2 \end{pmatrix} = P^{-1}\begin{pmatrix} E_1 & A_1 & B_1 \end{pmatrix}\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ G & F & R \end{pmatrix}}$$

Proposition 1. *If* (E_1, A_1, B_1) *is equivalent to* (E_2, A_2, B_2)*. Then* (E_1, B_1) *and* (A_1, B_1) *are feedback equivalent to* (E_2, B_2) *and* (A_2, B_2) *respectively.*

Corollary 1. *If necessary we can take a triple* (E, A, B) *where* (E, B) *(or* (A, B)*, but not both) in its Kronecker canonical form.*

Proposition 2. *Let (E, A, B) be a standardizable system and (E_1, A_1, B_1) a system equivalent to it. Then (E_1, A_1, B_1) is also standardizable.*

Proof. There exist matrices P, R invertible and G, G_0 such that

$$E = P^{-1} E_1 P + P^{-1} B_1 G$$
$$B = P^{-1} B_1 R$$

$E + BG_0$ invertible.

Then $\boxed{E_1 + B_1(GP^{-1} + RG_0P^{-1})}$ is invertible.

□

Theorem 1. *A system* $E\dot{x}(t) = Ax(t) + Bu(t)$ *may be standardized by a derivative feedback if, and only if, the pair of matrices* (E, B) *is either controllable or all its eigenvalues are non zero.*

Remark 3. Let $(E, B) \in M_n(\mathbb{C}) \times M_{n\times m}(\mathbb{C})$ be a pair of matrices.

$$\boxed{\begin{array}{c} \lambda_0 \in \mathbb{C} \text{ is an eigenvalue} \\ \Updownarrow \\ \text{rank}\begin{pmatrix}\lambda_0 I - E & B\end{pmatrix} < n \end{array}}$$

Example 2. We consider the following system

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

with

$$E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The unique eigenvalue of the pair (E, B) is $\boxed{\lambda = 1 \neq 0}$. Then, there exists a matrix G such that $E + BG$ is invertible.

We can take, for example $G = \begin{pmatrix} 1 & 0 \end{pmatrix}$, clearly

$$\boxed{\text{rank}\,(E + BG) = 2.}$$

Question:

If the system $E\dot{x}(t) = Ax(t) + Bu(t)$ is standardizable by means a derivative feedback, the standardized system

$$\dot{x}(t) = (E + BG)^{-1}Ax(t) + (E + BG)^{-1}Bu_1(t)$$

is controllable?

g-Controllability

Definition 4. *A triple of matrices* $(E, A, B) \in \mathcal{M}$ *is g-controllable if and only if the matrix*

$$\boxed{\begin{pmatrix} sE - A & B \end{pmatrix}}$$

has full rank for all $s \in \mathbb{C}$.

This definition is a generalization of controllability notion for pairs (A, B) of matrices representing standard systems.

Definition 5. *A pair* (A, B) *is controllable if and only if the matrix*

$$\boxed{\begin{pmatrix} sI - A & B \end{pmatrix}}$$

has full rank for all $s \in \mathbb{C}$.

Proposition 3. *The g-controllability character is invariant under the equivalence relation considered.*

Theorem 2. *Let $E\dot{x}(t) = Ax(t) + Bu(t)$ be a standardizable by means derivative feedback and g-controllable system. Then the standardized system is controllable.*

Remark 4. If a system is standardizable and g-controllable it is controllable in the sense of for any initial condition $x(0)$ we may choose a control such that the state response starting from $x(0)$ may arrive at any prescribed position.

Example 3. We consider the following system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

with

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The system is standardizable by means $G = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$. It is g-controllable:

$$\text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} = 3$$

for all $s \in \mathbb{C}$, then the standardized system is controllable:

$$\dot{x}(t) = A_1 x(t) + Bu(t), \tag{5}$$

with

$$A_1 = (E + BG)^{-1} A, \; B_1 = (E + BG)^{-1} B$$

$$\text{rank} \begin{pmatrix} B_1 & A_1 B_1 & {A_1}^2 B_1 \end{pmatrix} = 3$$

Remark 5. The system (**??**) is pole assignable, then there exists F such that the system

$$\dot{x}(t) = A_2 x(t) + B_1 u(t), \tag{6}$$

with $A_2 = A_1 + B_1 F$ has preassigned eigenvalues.

Taking $u_1(t) = u(t) + G\dot{x}(t) - Fx(t)$ in the system (**??**) $E\dot{x}(t) = Ax(t) + Bu(t)$, the standardized system is (**??**)

Controllability

Proposition 4. *A triple of matrices* $(E, A, B) \in \mathcal{M}$ *is controllable if and only it is standardizable and g-controllable, that is to say if:*

$$\begin{aligned} \text{rank} \begin{pmatrix} E & B \end{pmatrix} &= n \\ \text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} &= n, \ \textit{for all}\, s \in \mathbb{C}. \end{aligned}$$

Equivalent relation as Lie group action

$\mathcal{G} = Gl(n;\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times M_{m\times n}(\mathbb{C}) \times Gl(m;\mathbb{C})$ and its unit element by I. Note that $\mathcal{G}$ is a complex manifold.

Definition 6.

$$\alpha : \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M}$$

$$\alpha((P,F,G,R),(E,A,B)) =$$
$$(P^{-1}EP + P^{-1}BG, P^{-1}AP + P^{-1}BF, P^{-1}BR)$$

Equivalence relation induced: (E_1, A_1, B_1) and (E_2, A_2, B_2) are *equivalent* if and only if there exists $(P,F,G,R) \in \mathcal{G}$ such that

$$\boxed{\alpha((P,F,G,R),(E_1,A_2,B_1)) = (E_2,A_2,B_2)}$$

[1] V. I. Arnold, *On matrices depending on parameters*, Russian Math. Surveys, 26: 2 (1971), pp. 29–43.

- **Orbit**

$$\boxed{\mathcal{O}(E,A,B) = \alpha(\mathcal{G} \times (E,A,B))}$$

- **Tangent space**

$T_{(E,A,B)}\mathcal{O}(E,A,B) =$
$\{(EP - PE + BV, AP - PA + BU, BR - PB)\}$

for all $P \in M_n(\mathbb{C})$, $U, V \in M_{m\times n}(\mathbb{C})$, and $R \in M_m(\mathbb{C})$.

- **Normal space**

$$T_{(E,A,B)}\mathcal{O}(E,A,B)^{\perp} = \{(X,Y,Z) : \left.\begin{array}{rl} \overline{X}^t E - E\overline{X}^t + \overline{Y}^t A - A\overline{Y}^t - B\overline{Z}^t & = 0 \\ \overline{Z}^t B = 0, \overline{X}^t B = 0, \overline{Y}^t B & = 0 \end{array}\right\}$$

- **Differentiable families of triples**

$$\begin{array}{rcl} \mathcal{T}:\mathcal{U} & \longrightarrow & \mathcal{M} \\ \lambda & \longrightarrow & (E(\lambda), A(\lambda), B(\lambda)) \\ 0 & \longrightarrow & (E, A, B) \end{array}$$

$\mathcal{U} \subset \mathbb{R}^{\ell}$.

- **Generic differentiable families of triples**

$$\boxed{\mathcal{T}(\lambda) = (E, A, B) + H}$$

with H transversal to $T_{(E,A,B)}\mathcal{O}(E, A, B)$.

- For example

$$\boxed{\mathcal{T}(\lambda) = (E, A, B) + T_{(E,A,B)}\mathcal{O}(E, A, B)^{\perp}}$$

Closure hierarchy of orbits

- **Frontier condition:**

Proposition 5. *Let* (E_0, A_0, B_0) *in the closure* $\overline{\mathcal{O}(E, A, B)}$ *of* $\mathcal{O}(E, A, B)$.

Then $\mathcal{O}(E_0, A_0, B_0) \subset \overline{\mathcal{O}(E, A, B)}$.

Corollary 2. $(E_0, A_0, B_0) \in \overline{\mathcal{O}(E, A, B)}$. *Then* $\overline{\mathcal{O}(E_0, A_0, B_0)} \subset \overline{\mathcal{O}(E, A, B)}$.

◇ Problem:

There are a infinite number of orbits.

Strata

Definition 7. *Stratum = Union of orbits having the same set of discrete invariants.*

- Case $n = 2$, $m = 1$.

$$\mathcal{S}_0 = \bigcup_{a_i \in \mathbb{C}} \mathcal{O}\left(\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), \left(\begin{array}{cc} a_1 & a_2 \\ 0 & 0 \end{array}\right)\left(\begin{array}{c} 0 \\ 1 \end{array}\right)\right)$$

$$\mathcal{S}_1 = \bigcup_{e \in \mathbb{C}} \mathcal{O}\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right)\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_2^1 = \bigcup_{e,a \in \mathbb{C}} \mathcal{O}\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right)\left(\begin{array}{c} 1 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_2^2 = \bigcup_{\substack{e_i, a_i \in \mathbb{C} \\ e_1 \neq e_2}} \mathcal{O}\left(\left(\begin{array}{cc} e_1 & 0 \\ 0 & e_2 \end{array}\right), \left(\begin{array}{cc} a_1 & 1 \\ a_2 & a_3 \end{array}\right)\left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_3^1 = \bigcup_{\substack{e, a_i \in \mathbb{C} \\ a_2 \neq 0}} \mathcal{O}\left(\left(\begin{array}{cc} e & 1 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} a_1 & 0 \\ a_2 & a_3 \end{array}\right)\left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_3^2 = \bigcup_{\substack{e_i, a_i \in \mathbb{C} \\ e_1 \neq e_2, \\ a_1 \neq a_3}} \mathcal{O}\left(\left(\begin{array}{cc} e_1 & 0 \\ 0 & e_2 \end{array}\right), \left(\begin{array}{cc} a_1 & 1 \\ 0 & a_3 \end{array}\right)\left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_4^1 = \bigcup_{\substack{e,\, a_i \in \mathbb{C} \\ a_1 \neq a_2}} \mathcal{O}\left(\left(\begin{array}{cc} e & 1 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_4^2 = \bigcup_{\substack{e_i,\, a \in \mathbb{C} \\ e_1 \neq e_2}} \mathcal{O}\left(\left(\begin{array}{cc} e_1 & 0 \\ 0 & e_2 \end{array}\right), \left(\begin{array}{cc} a & 1 \\ 0 & a \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_5^1 = \bigcup_{\substack{e,\, a_i \in \mathbb{C} \\ a_1 \neq a_2}} \mathcal{O}\left(\left(\begin{array}{cc} e & 0 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} a_1 & 0 \\ 0 & a_2 \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_5^2 = \bigcup_{\substack{e_i,\, a \in \mathbb{C} \\ e_1 \neq e_2}} \mathcal{O}\left(\left(\begin{array}{cc} e_1 & 0 \\ 0 & e_2 \end{array}\right), \left(\begin{array}{cc} a & 0 \\ 0 & a \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_6^1 = \bigcup_{e,a\in\mathbb{C}} \mathcal{O}\left(\left(\begin{array}{cc} e & 0 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} a & 1 \\ 0 & a \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_6^2 = \bigcup_{e,a\in\mathbb{C}} \mathcal{O}\left(\left(\begin{array}{cc} e & 1 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} a & 0 \\ 0 & a \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\mathcal{S}_8 = \bigcup_{e,a\in\mathbb{C}} \mathcal{O}\left(\left(\begin{array}{cc} e & 0 \\ 0 & e \end{array}\right), \left(\begin{array}{cc} a & 0 \\ 0 & a \end{array}\right) \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right)$$

$$\boxed{\operatorname{codim} \mathcal{S}_i = i}$$

• The collection of strata is a partition of the space $\mathcal{M}$:

$$\bigcup \mathcal{S}_i = \mathcal{M}$$

$$\mathcal{S}_i \cap \mathcal{S}_j = \emptyset \text{ if } i \neq j.$$

• Frontier condition:

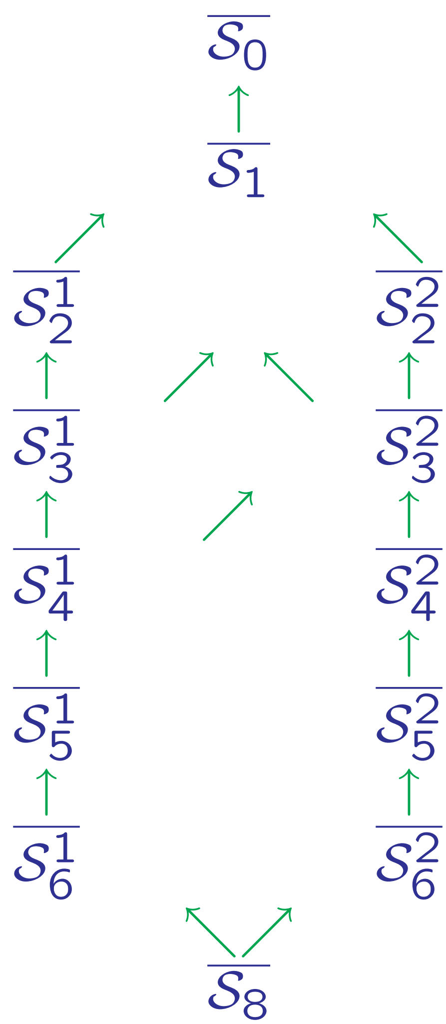
Proposition 6. *Let* (E_0, A_0, B_0) *in the closure* $\overline{\mathcal{S}(E, A, B)}$ *of* $\mathcal{S}(E, A, B)$.

Then $\mathcal{S}(E_0, A_0, B_0) \subset \overline{\mathcal{S}(E, A, B)}$.

Corollary 3. $(E_0, A_0, B_0) \in \overline{\mathcal{S}(E, A, B)}$. *Then* $\overline{\mathcal{S}(E_0, A_0, B_0)} \subset \overline{\mathcal{S}(E, A, B)}$.

• There are a finite number of strata

Closure hierarchy



- Strata of Standardizable triples:

$$\boxed{\mathcal{S}tand \subset \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2^1}$$

- Strata of Controllable triples:

$$\boxed{\mathcal{C}ont \subset \mathcal{S}tand \subset \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2^1}$$

- Subtratification of $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2^1$

◇ Partition of $\mathcal{S}_0$

$$\tilde{\mathcal{S}}_{0_1} = \bigcup_{\substack{a_i \in \mathbb{C} \\ a_1 \neq 0}} \mathcal{O}\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_1 & a_2 \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$$

$$\tilde{\mathcal{S}}_{0_2} = \bigcup_{\substack{a_i \in \mathbb{C} \\ a_1 = 0}} \mathcal{O}\left(\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_1 & a_2 \\ 0 & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\right)$$

$$\tilde{\mathcal{S}}_{0_1} \bigcup \tilde{\mathcal{S}}_{0_2} = \mathcal{S}_0, \qquad \tilde{\mathcal{S}}_{0_1} \bigcap \tilde{\mathcal{S}}_{0_2} = \emptyset$$

$\diamond$ Partition of $\mathcal{S}_1$

$$\tilde{\mathcal{S}}_{1_1} = \bigcup_{\substack{e \in \mathbb{C} \\ e \neq 0}} \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$\tilde{\mathcal{S}}_{1_2} = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$\tilde{\mathcal{S}}_{1_1} \bigcup \tilde{\mathcal{S}}_{1_2} = \mathcal{S}_1, \qquad \tilde{\mathcal{S}}_{1_1} \bigcap \tilde{\mathcal{S}}_{1_2} = \emptyset$$

$\diamond$ Partition of $\mathcal{S}_2^1$

$$\tilde{\mathcal{S}}_{2_1}^1 = \bigcup_{\substack{e, a \in \mathbb{C} \\ e \neq 0}} \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$\tilde{\mathcal{S}}_{2_2}^1 = \mathcal{O}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

$$\tilde{\mathcal{S}}_{2_1}^1 \bigcup \tilde{\mathcal{S}}_{2_2}^1 = \mathcal{S}_2^1, \qquad \tilde{\mathcal{S}}_{2_1}^1 \bigcap \tilde{\mathcal{S}}_{2_2}^1 = \emptyset$$

Proposition 7. *The standardizable set is*

$$\boxed{\mathcal{S}tand = \mathcal{S}_0 \bigcup \tilde{\mathcal{S}}_{1_1} \bigcup \tilde{\mathcal{S}}^1_{2_1}}$$

Proof.

$$\begin{array}{l} \text{rank} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \text{rank} \begin{pmatrix} 0 & 0 & 1 \\ 0 & e & 0 \end{pmatrix} = 2 \iff e \neq 0 \end{array}$$

□

Proposition 8. *The controllable set is*

$$\boxed{\mathcal{C}ont = \tilde{\mathcal{S}}_{0_1} \bigcup \tilde{\mathcal{S}}_{1_1}}$$

Proof.

$$\begin{array}{l} \text{rank} \begin{pmatrix} -a_1 & -a_2+s & 0 \\ 0 & 0 & 1 \end{pmatrix} = 2 \implies a_1 \neq 0 \\ \text{rank} \begin{pmatrix} 0 & 0 & 1 \\ -1 & se & 0 \end{pmatrix} = 2 \ \forall e \\ \text{rank} \begin{pmatrix} 0 & 0 & 1 \\ 0 & a-se & 0 \end{pmatrix} = 1 \text{ for some} s \in \mathbb{C} \end{array}$$

□

The structure in the neighborhood of strata

$\widetilde{\mathcal{S}}_{0_1}$ is an open set,

$\widetilde{\mathcal{S}}_{0_2}$: $\boxed{\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x & a_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}$,

$\widetilde{\mathcal{S}}_{1_1}$: $\boxed{\left(\begin{pmatrix} 0 & 0 \\ x & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}$

$\widetilde{\mathcal{S}}_{1_2}$: $\boxed{\left(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}$

$\widetilde{\mathcal{S}}^1_{2_1}$: $\boxed{\left(\begin{pmatrix} 0 & 0 \\ x & e \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ y & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}$

$\widetilde{\mathcal{S}}^1_{2_2}$: $\boxed{\left(\begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ z & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}$

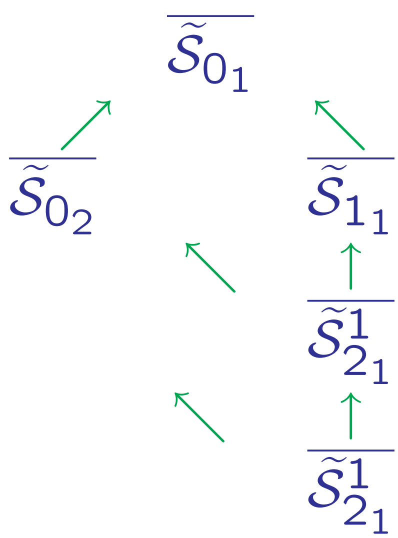
Closure hierarchy

$$\text{codim}\widetilde{\mathcal{S}}_{0_1} = 0 \qquad \boxed{\overline{\widetilde{\mathcal{S}}_{0_2}} \subset \overline{\widetilde{\mathcal{S}}_{0_1}} \subseteq \overline{\widetilde{\mathcal{S}}_{0}}}$$
$$\text{codim}\widetilde{\mathcal{S}}_{0_2} = 1$$

$$\text{codim}\widetilde{\mathcal{S}}_{1_1} = 1 \qquad \boxed{\overline{\widetilde{\mathcal{S}}_{1_2}} \subset \overline{\widetilde{\mathcal{S}}_{1_1}} \subseteq \overline{\widetilde{\mathcal{S}}_{1}}}$$
$$\text{codim}\widetilde{\mathcal{S}}_{1_2} = 2$$

$$\text{codim}\widetilde{\mathcal{S}}^1_{2_1} = 2 \qquad \boxed{\overline{\widetilde{\mathcal{S}}^1_{2_2}} \subset \overline{\widetilde{\mathcal{S}}^1_{2_1}} \subseteq \overline{\widetilde{\mathcal{S}}^1_{2}}}$$
$$\text{codim}\widetilde{\mathcal{S}}^2_{2_2} = 3$$

Closure hierarchy



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